# GROWTH OF HILBERT COEFFICIENTS OF SYZYGY MODULES

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Abstract. Let $(A,\mathfrak{m})$ be a complete intersection ring of dimension $d$ and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a maximal Cohen-Macaulay $A$-module. For $i = 0, 1, \cdots, d$, let $e_i^I(M)$ denote the $i^{th}$ Hilbert -coefficient of $M$ with respect to $I$. We prove that for $i = 0, 1, 2$, the function $j \mapsto e_i^I(\mathrm{Syz}_j^A(M))$ is of quasi-polynomial type with period 2. Let $G_I(M)$ be the associated graded module of $M$ with respect to $I$. If $G_I(A)$ is Cohen-Macaulay and $\dim A \leq 2$ we also prove that the functions $j \mapsto \operatorname{depth} G_I(\mathrm{Syz}_{2j+i}^A(M))$ are eventually constant for $i = 0, 1$. Let $\xi_I(M) = \lim_{l\to\infty} \operatorname{depth} G_{I^l}(M)$. Finally we prove that if $\dim A = 2$ and $G_I(A)$ is Cohen-Macaulay then the functions $j \mapsto \xi_I(\mathrm{Syz}_{2j+i}^A(M))$ are eventually constant for $i = 0, 1$.

## 1. introduction

Let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension $d$ and let $M$ be a finitely generated $A$-module of dimension $r$. Let $I$ be an $\mathfrak{m}$-primary ideal. Let $\ell(N)$ denote the length of an $A$-module $N$. The function $H_I^{(1)}(M,n) = \ell(M/I^{n+1}M)$ is called the *Hilbert-Samuel* function of $M$ with respect to $I$. It is well-known that there exists a polynomial $P_I(M,X) \in \mathbb{Q}[X]$ of degree $r$ such that $P_I(M,n) = H_I^{(1)}(M,n)$ for $n \gg 0$. The polynomial $P_I(M,X)$ is called the Hilbert-Samuel polynomial of $M$ with respect to $I$. We write

$$P_I(M,X) = \sum_{i=0}^{r} (-1)^i e_i^I(M) \binom{X+r-i}{r-i}.$$

The integers $e_i^I(M)$ are called the $i^{th}$-Hilbert coefficent of $M$ with respect to $I$. The zeroth Hilbert coefficent $e_0^I(M)$ is called the *multiplicity* of $M$ with respect to $I$.

For $j \geq 0$ let $\mathrm{Syz}_j^A(M)$ denote the $j^{th}$ syzygy of $M$. In this paper we investigate the function $j \mapsto e_i^I(\mathrm{Syz}_j^A(M))$ for $i \geq 0$. It becomes quickly apparent that for reasonable answers we need that the minimal resolution of $M$ should have some structure. Minimal resolutions of modules over complete intersection rings have a good structure. If $A = B/(f_1, \cdots, f_c)$ with $\mathbf{f} = f_1, \cdots, f_c$ a $B$-regular sequence and $\operatorname{projdim}_B M$ is finite then also the minimal resolution of $M$ has a nice structure. The definitive class of modules with a good structure theory of their minimal resolution is the class of modules with finite complete intersection dimension, see [2]. We are able to prove our results for a more restrictive class of modules than modules of finite CI-dimension.

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*Date*: January 30, 2015.

1991 *Mathematics Subject Classification.* Primary 13D40; Secondary 13A30.

**Definition 1.1.** We say the $A$ module $M$ has finite GCI-dimension if there is a flat local extension $(B, \mathfrak{n})$ of $A$ such that

(1) $\mathfrak{m}B = \mathfrak{n}$.
(2) $B = Q/(f_1, \cdots, f_c)$ , where $Q$ is local and $f_1, \ldots, f_c$ is a $Q$-regular sequence.
(3) $\operatorname{projdim}_Q M \otimes_A B$ is finite.

We note that every finitely generated module over an abstract complete intersection ring has finite GCI dimension. If $A = R/(f_1, \cdots, f_c)$ with $f_1, \cdots, f_c$ a $R$-regular sequence and $\operatorname{projdim}_R M$ is finite then also $M$ has finite GCI-dimension. We also note that if $M$ has finite GCI dimesnion then it has finite CI-dimension. If $M$ has finite CI-dimension then the function $i \mapsto \ell(\operatorname{Tor}_i^A(M, k))$ is of quasi-polynomial type with degree two. Set $\operatorname{cx}(M) =$ degree of this function $+1$. (See 2.7 for degree of a function of quasi-polynomial type).

Let $G_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of $A$ with respect to $I$. Let $G_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1}M$ be the associated graded module of $M$ with respect to $I$. Our main result is

**Theorem 1.2.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$ and let $M$ be a maximal Cohen-Macaulay $A$-module. Let $I$ be an $\mathfrak{m}$-primary ideal. Assume $M$ has finite GCI dimension. Then for $i = 0, 1, 2$, the function $j \mapsto e_i^I(\operatorname{Syz}_j^A(M))$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$.*

Next we consider the asymptotic behavior of depth of associated graded modules of syzygy modules. We prove

**Theorem 1.3.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $\leq 2$ and let $M$ be a maximal Cohen-Macaulay $A$-module. Let $I$ be an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Assume $M$ has finite GCI dimension. Then the functions $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}_{2j}^A(M))$ and $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}_{2j+1}^A(M))$ are constant for $j \gg 0$.*

If $M$ is a Cohen-Macaulay $A$-module and $I$ is $\mathfrak{m}$-primary then it is known that $\operatorname{depth} G_{I^s}(M)$ is constant for all $s \gg 0$, see [6, 2.2] (also see [10, 7.6]). Set $\xi_I(M) = \lim \operatorname{depth} G_{I^s}(M)$. We prove the following:

**Theorem 1.4.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 and let $M$ be a maximal Cohen-Macaulay $A$-module. Let $I$ be an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Assume $M$ has finite GCI dimension. Then the functions $j \mapsto \xi_I(\operatorname{Syz}_{2j}^A(M))$ and $j \mapsto \xi_I(\operatorname{Syz}_{2j+1}^A(M))$ are constant for $j \gg 0$.*

**1.5.** *Dual Hilbert-Samuel function:* Assume $A$ has a canonical module $\omega$. The function $D^I(M, n) = \ell(\operatorname{Hom}_A(M, \omega/I^{n+1}\omega)$ is called the *dual Hilbert-Samuel* function of $M$ with respect to $I$. In [12] it is shown that there exist a polynomial $t^I(M, z) \in \mathbb{Q}[z]$ of degree $d$ such that $t^I(M, n) = D^I(M, n)$ for all $n \gg 0$. We write

$$t^I(M, X) = \sum_{i=0}^{d} (-1)^i c_i^I(M) \binom{X + r - i}{r - i}.$$

The integers $c_i^I(M)$ are called the $i^{th}$- dual Hilbert coefficient of $M$ with respect to $I$. The zeroth dual Hilbert coefficient $c_0^I(M)$ is equal to $e_0^I(M)$, see [12, 2.5]. We prove:

**Theorem 1.6.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$, with a canonical module $\omega$ and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a maximal Cohen-Macaulay $A$-module. Assume $M$ has finite GCI dimension. Then for $i = 0, 1, \cdots, d$*

*the function $j \mapsto c_i^I(\operatorname{Syz}_j^A(M))$ is of quasi-polynomial type with period two. If $A$ is a complete intersection then the degree of each of the above functions $\leq \operatorname{cx}(M)-1$.*

Although Theorem 1.6 looks more complicated than Theorem 1.2, its proof is considerably simpler.

Let $H^i(-)$ denote the $i^{th}$ local cohomology functor of $G_I(A)$ with respect to $G_I(A)_+ = \bigoplus_{n>0} I^n/I^{n+1}$. Set

$$\operatorname{reg}(G_I(M)) = \max\{i+j \mid H^i(G_I(M))_j \neq 0\},$$

the *regularity* of $G_I(M)$. Set

$$a_i(G_I(M)) = \max\{j \mid H^i(G_I(M))_j \neq 0\}.$$

Assume that the residue field of $A$ is infinite. Let $J$ be a minimal reduction of $I$. Say $I^{r+1} = JI^r$. Let $M$ be a maximal Cohen-Macaulay $A$-module. Then it is well-known that $a_d(G_I(M)) \leq r-d$; see [13, 3.2](also see [3, 18.3.12]). We prove

**Theorem 1.7.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ and let $M$ be a maximal Cohen-Macaulay $A$-module. Let $I$ be an $\mathfrak{m}$-primary ideal. Assume $M$ has finite GCI dimension. Then the set*

$$\left\{ \frac{a_{d-1}(G_I(\operatorname{Syz}_i^A(M))}{i^{\operatorname{cx}(M)-1}} \right\}_{i\geq 1}$$

*is bounded.*

Here is an overview of the contents of this paper. In section two we discuss many preliminary facts that we need. In section three we introduce a technique which is useful to prove our results. In section four(five) we discuss the case when dimension of $A$ is one(two) respectively. The proof of Theorem 1.2 is divided in these two sections. The proof of Theorem 1.3 is in section five. In section six we prove Theorem 1.4. Theorem 1.6 is proved in section seven. Finally in section eight we prove Theorem 1.7.

## 2. Preliminaries

In this section we collect a few preliminary results that we need.

**2.1.** The *Hilbert function* of $M$ with respect to $I$ is the function

$$H_I(M,n) = \lambda(I^nM/I^{n+1}M) \quad \text{for all } n \geq 0.$$

It is well known that the formal power series $\sum_{n\geq 0} H_I(M,n)z^n$ represents a rational function of a special type:

$$\sum_{n\geq 0} H_I(M,n)z^n = \frac{h_I(M,z)}{(1-z)^r} \quad \text{where } r = \dim M \text{ and } h_I(M,z) \in \mathbb{Z}[z].$$

It can be shown that $e_i^I(M) = (h_I(M,z))^{(i)}(1)/i!$ for all $0 \leq i \leq r$. It is convenient to set $e_i^I(M) = (h_I(M,z))^{(i)}(1)/i!$ even when $i \geq r$. The number $e_0^I(M)$ is also called the *multiplicity* of $M$ with respect to $I$.

**I:** *Superficial elements.*

For definition and basic properties of superficial sequences see [8, p. 86-87]. The following result is well-known to experts. We give a proof due to lack of a reference.

**Lemma 2.2.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let $I$ be an $\mathfrak{m}$-primary ideal. Assume the residue field $k = A/\mathfrak{m}$ is uncountable. Let $\{M_n\}_{n\geq 1}$ be a sequence of maximal Cohen-Macaulay $A$-modules. Then there exists $\mathbf{x} = x_1, \ldots, x_d \in I$ such that $\mathbf{x}$ is an $M_n$-superficial sequence (with respect to $I$) for all $n \geq 1$.*

*Proof.* We recall the construction of a superficial element for a finitely generated module $M$ of dimension $r \geq 1$ (note $M$ need not be Cohen-Macaulay). Let $\mathcal{M}$ be the maximal ideal of $G_I(A)$. Let

$$\operatorname{Ass}^*(G_I(M)) = \{P \mid P \in \operatorname{Ass}(G_I(M)) \text{ and } P \neq \mathcal{M}\}.$$

Set $L = I/I^2$ and $V = L/\mathfrak{m}L = I/\mathfrak{m}I$. For $P \in \operatorname{Spec} G_I(A)$ let $P_1 = P \cap L$ and $\overline{P_1} = (P_1 + \mathfrak{m}I)/\mathfrak{m}I$. If $P \neq \mathcal{M}$ then $P_1 \neq L$. By Nakayama's Lemma it follows easily that $\overline{P_1} \neq V$. If $k$ is infinite then

$$S(M) = V \setminus \bigcup_{P \in \operatorname{Ass}^*(G_I(M))} \overline{P_1} \neq \emptyset.$$

Then $u \in I$ such that $\overline{u} \in S(M)$ is an $M$-superficial element with respect to $I$.

Now assume that $k$ is uncountable and let $\mathcal{F} = \{M_n\}_{n\geq 1}$ be a sequence of maximal Cohen-Macaulay $A$-modules. Set $M_0 = A$. Then

$$S(\mathcal{F}) = V \setminus \bigcup_{n\geq 0} \bigcup_{P \in \operatorname{Ass}^*(G_I(M_n))} \overline{P_1} \neq \emptyset.$$

If $u \in I$ such that $\overline{u} \in S(\mathcal{F})$ then $\overline{u} \in S(M_n)$ for all $n \geq 1$. Thus $u$ is $M_n$-superficial element with respect to $I$ for all $n \geq 0$. Set $x_1 = u$. If $d \geq 2$ then note that $\{M_n/x_1M_n\}$ is a sequence of maximal Cohen-Macaulay $A/(x_1)$ modules. So by induction the result follows. □

**2.3. Associated graded module and Hilbert function mod a superficial element:**
Let $x \in I$ be $M$-superficial. Set $N = M/xM$. Let $r = \dim M$. The following is well-known cf., [8].

(1) Set $b_I(M, z) = \sum_{i\geq 0} \ell\big((I^{n+1}M \colon_M x)/I^nM\big)z^n$. Since $x$ is $M$-superficial we have $b_I(M, z) \in \mathbb{Z}[z]$.
(2) $h_I(M, z) = h_I(N, z) - (1 - z)^r b_I(M, z)$; cf., [8, Corollary 10 ].
(3) So we have
   (a) $e_i(M) = e_i(N)$ for $i = 0, \ldots, r - 1$.
   (b) $e_r(M) = e_r(N) - (-1)^r b_I(M, 1)$.
(4) The following are equivalent
   (a) $x^*$ is $G_I(M)$-regular.
   (b) $G_I(N) = G_I(M)/x^*G_I(M)$
   (c) $b_I(M, z) = 0$
   (d) $e_r(M) = e_r(N)$.
(5) *(Sally descent)* If $\operatorname{depth} G_I(N) > 0$ then $x^*$ is $G_I(M)$-regular.

**II:***Base change.*

**2.4.** Let $\phi\colon (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be a flat local ring homomorphism with $\mathfrak{m}A' = \mathfrak{m}'$. Set $I' = IA'$ and if $N$ is an $A$-module set $N' = N \otimes A'$. It can be easily seen that

(1) $\lambda_A(N) = \lambda_{A'}(N')$.
(2) $\operatorname{projdim}_A N = \operatorname{projdim}_{A'} N'$.

(3) $H^I(M, n) = H^{I'}(M', n)$ for all $n \geq 0$.
(4) $\dim M = \dim M'$ and $\operatorname{grade}(K, M) = \operatorname{grade}(KA', M')$ for any ideal $K$ of $A$.
(5) $\operatorname{depth} G_I(M) = \operatorname{depth} G_{I'}(M')$.

The specific base changes we do are the following:

(i) If $M$ is a maximal Cohen-Macaulay $A$-module with finite GCI dimension then by definition there exists a flat homomorphism $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ with $\mathfrak{m}B = \mathfrak{n}$ and $B = Q/(f_1, \ldots, f_c)$ for some local ring $Q$ and a $Q$-regular sequence $f_1, \ldots, f_c$ such that $\operatorname{projdim}_Q M \otimes_A B < \infty$. In this case we set $B = A'$.

(ii) If $k \subseteq k'$ is an extension of fields then it is well-known that there exists a flat local ring homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ with $\mathfrak{m}A' = \mathfrak{m}'$ and $A'/\mathfrak{m}' = k'$. We use this construction when the residue field $k$ of $A$ is finite or countably infinite. We take $k'$ to be any uncountable field containing $k$.

(iii) We can also take $A'$ to be the completion of $A$.

**Remark 2.5.** Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and let $M$ be a maximal Cohen-Macaulay $A$-module with finite GCI dimension. After doing the base-changes (i), (ii) and (iii) above we may assume that $A = Q/(f_1, \ldots, f_c)$ for some *complete* Cohen-Macaulay local ring $Q$ and a $Q$-regular sequence $f_1, \ldots, f_c$ such that $\operatorname{projdim}_Q M < \infty$. Furthermore we may assume that the residue field of $A$ is uncountable.

**III:** *Quasi-polynomial functions of period 2.*

Let us recall that a function $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is said to be of *quasi-polynomial type* with period $g \geq 1$ if there exists polynomials $P_0, P_1, \ldots, P_{g-1} \in \mathbb{Q}[X]$ such that $f(mg + i) = P_i(m)$ for all $m \gg 0$ and $i = 0, \cdots, g-1$.

We need the following well-known result regarding quasi-polynomials of period two.

**Lemma 2.6.** *Let $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. The following are equivalent:*

(i) *$f$ is of quasi-polynomial type with period 2.*
(ii) *$\sum_{n \geq 0} f(n) z^n = \frac{h(z)}{(1-z^2)^c}$ for some $h(z) \in \mathbb{Z}[z]$ and $c \geq 0$.*

*Furthermore if $P_0, P_1 \in Q[X]$ are polynomials such that $f(2m + i) = P_i(m)$ for all $m \gg 0$ and $i = 0, 1$ then $\deg P_i \leq c - 1$.* $\square$

*Convention:* We set the degree of the zero-polynomial to be $-\infty$.

**Remark 2.7.** Let $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be of quasi-polynomial type with period 2. Let $P_0, P_1 \in Q[X]$ be polynomials such that $f(2m + i) = P_i(m)$ for all $m \gg 0$ and $i = 0, 1$. Then set $\deg f = \max\{\deg P_0, \deg P_1\}$.

As a consequence we get that

**Corollary 2.8.** *Let $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. If the function $g(n) = f(n) + f(n-1)$ is of quasi-polynomial type with period 2 then so is $f$. Furthermore $\deg f = \deg g$.*

*Sketch of a proof.* Set $u(z) = \sum_{n \geq 0} f(n) z^n$. Then by our hypothesis and Lemma 2.6 we have

$$(1 + z)u(z) + f(0)z = \frac{h(z)}{(1 - z^2)^c},$$

for some $h(z) \in \mathbb{Z}[z]$ and $c \geq 0$. An easy computation now shows that

$$u(z) = \frac{r(z)}{(1 - z^2)^c},$$

for some $r(z) \in \mathbb{Z}[z]$. Thus $f$ is of quasi-polynomial type with period two.

As $f(n) \leq g(n)$ we clearly get $\deg f \leq \deg g$. As $g(2n) = f(2n) + f(2n-1)$ and $g(2n+1) = f(2n+1) + f(2n)$ we get that $\deg g \leq \deg f$. $\square$

The following result is also well-known:

**Lemma 2.9.** *Let $f\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that*

$$\sum_{m,n\geq 0} f(m,n) z^m w^n = \frac{h(z,w)}{(1-z^2)^c(1-w)^d}$$

*Then for $m,n \gg 0$ we have*

$$f(m,n) = \sum_{i=0}^{d-1} (-1)^i e_i(m) \binom{n+d-1-i}{d-1-i},$$

*where the functions $m \mapsto e_i(m)$ for $i = 0,\ldots,d-1$ are of quasi-polynomial type with period 2 and degree $\leq c-1$.* $\square$

**IV:** *Eisenbud operators*

Let $Q$ be a Noetherian ring and let $\mathbf{f} = f_1, \ldots f_c$ be a regular sequence in $Q$. Set $A = Q/(\mathbf{f})$. Let $M$ be a finitely generated $A$-module with $\operatorname{projdim}_Q M$ finite.

**2.10.** Let $\mathbb{F}: \cdots F_n \to \cdots F_1 \to F_0 \to 0$ be a free resolution of $M$ as a $A$-module.

Let $t_1, \ldots t_c\colon \mathbb{F}(+2) \to \mathbb{F}$ be the *Eisenbud-operators*; see [5, section 1.]. Then

(1) $t_i$ are uniquely determined up to homotopy.

(2) $t_i, t_j$ commute up to homotopy.

Let $T = A[t_1, \ldots, t_c]$ be a polynomial ring over $A$ with variables $t_1, \ldots, t_c$ of degree 2. Let $D$ be an $A$-module. The operators $t_j$ give well-defined maps

$$t_j\colon \operatorname{Ext}^i_A(M,D) \to \operatorname{Ext}^{i+2}_R(M,D) \quad \text{for } 1 \leq j \leq c \text{ and all } i,$$
$$t_j\colon \operatorname{Tor}^{i+2}_A(M,D) \to \operatorname{Tor}^i_R(M,D) \quad \text{for } 1 \leq j \leq c \text{ and all } i.$$

This turns $\operatorname{Ext}^*_A(M,D) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(M,D)$ and $\operatorname{Tor}^*_A(M,D) = \bigoplus_{i\geq 0} \operatorname{Tor}^i_A(M,D)$ into modules over $T$ (here we give an element $t \in \operatorname{Tor}^i_A(M,D)$ degree $-i$). Furthermore these structure depend only on $\mathbf{f}$, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

**2.11.** Gulliksen, [7, 3.1], proved that if $\operatorname{projdim}_Q M$ is finite then $\operatorname{Ext}^*_A(M,D)$ is a finitely generated $T$-module. If $A$ is local and $D = k$, the residue field of $A$, Avramov in [1, 3.10] proved a converse; i.e., if $\operatorname{Ext}^*_A(M,k)$ is a finitely generated $T$-module then $\operatorname{projdim}_Q M$ is finite. For a more general result, see [2, 4.2].

**Definition 2.12.** (with notation as above:) Assume $A$ is local with residue field $k$. Set $\operatorname{cx} M = \dim_T \operatorname{Ext}^*_A(M,k)$, the *complexity* of $M$.

We need the following result regarding the growth of lengths of certain Tor's. Recall a graded module $X$ over $T = A[t_1, \ldots, t_c]$ is said to be *-Artinian $T$-module if every descending chain of graded submodules of $X$ terminates.

**Proposition 2.13.** *Let $(Q, \mathfrak{n})$ be a complete Noetherian ring and let $\mathbf{f} = f_1, \ldots f_c$ be a regular sequence in $Q$. Set $A = Q/(\mathbf{f})$ and $\mathfrak{m} = \mathfrak{n}/(\mathbf{f})$. Let $M$ be a finitely generated $A$-module with* $\operatorname{projdim}_Q M$ *finite. Let $D$ be a non-zero $A$-module of finite length. Let $\operatorname{Ext}^*_A(M,D)$ be a finitely generated $T = A[t_1, \ldots, t_c]$-module as above. Then*

(1) $\dim_T \operatorname{Ext}^*_A(M, D) \leq \operatorname{cx}(M)$.
(2) *The function* $n \mapsto \ell(\operatorname{Ext}^n_A(M, D))$ *is of quasi-polynomial type with period* 2 *and degree* $\leq \operatorname{cx}(M) - 1$.
(3) $\operatorname{Tor}^*_A(M, D)$ *is a *-Artinian* $T$*-module. Here* $t \in \operatorname{Tor}^A_n(M, D)$ *has degree* $-n$.
(4) *The function* $n \mapsto \ell(\operatorname{Tor}^n_A(M, D))$ *is of quasi-polynomial type with period* 2 *and degree* $\leq \operatorname{cx}(M) - 1$.

*Proof.* (1) This follows from [2, Theorem 5.3].

(2) This easily follows from (1).

For (3), (4) we use the following result. Let $E$ be the injective hull of $k$. Then we have

$$(\dagger) \qquad \operatorname{Hom}_A(\operatorname{Tor}^A_n(M, D), E) \cong \operatorname{Ext}^n_A(M, \operatorname{Hom}_A(D, E)).$$

(3) From (†) it follows that the Matlis dual of $\operatorname{Tor}^*_A(M, D)$ is $\operatorname{Ext}^*_A(M, \operatorname{Hom}_A(D, E))$. By Gulliksen's result we have that $\operatorname{Ext}^*_A(M, \operatorname{Hom}_A(D, E))$ is a finitely generated graded $T$-module. So by Matlis-duality [4, 3.6.17] we get that $\operatorname{Tor}^*_A(M, D)$ is a *-Artinian $T$-module.

(4) This follows from (2) and (†). $\square$

As an immediate consequence we obtain:

**Corollary 2.14.** *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d$ *and let* $I$ *be an* $\mathfrak{m}$*-primary ideal. Let* $M$ *be a maximal Cohen-Macaulay* $A$*-module. If* $M$ *has finite GCI dimension over* $A$ *then the function* $i \mapsto e^I_0(\operatorname{Syz}^A_i(M))$ *is of quasi-polynomial type with period* 2 *and degree* $= \operatorname{cx}(M) - 1$.

*Proof.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with infinite residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Then $\operatorname{rank} F_i = \ell(\operatorname{Tor}^A_i(M, k))$. So by 2.13 we get that the function $i \mapsto \operatorname{rank} F_i$ is quasi-polynomial of period two. Set $M_i = \operatorname{Syz}^A_i(M)$. The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ yields $e^I_0(M_i) + e^I_0(M_{i+1}) = (\operatorname{rank} F_i) e^I_0(A)$. The result now follows from 2.8. $\square$

We also prove:

**Proposition 2.15.** *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d$ *and let* $I$ *be an* $\mathfrak{m}$*-primary ideal. Let* $M$ *be a maximal Cohen-Macaulay* $A$*-module. Assme* $M$ *has finite GCI dimension over* $A$*. Fix* $n \geq 0$*. Then the function*

$$i \mapsto \ell\left(\frac{I^n \operatorname{Syz}^i_A(M)}{I^{n+1} \operatorname{Syz}^i_A(M)}\right)$$

*is of quasi-polynomial type with period two and degree* $\leq \operatorname{cx}(M) - 1$.

*Proof.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with infinite residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}^A_i(M)$. Fix $n \geq 0$. The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ yields an exact sequence

$$0 \to \operatorname{Tor}^A_{i+1}(M, A/I^{n+1}) \to M_{i+1}/I^{n+1}M_{i+1} \to F_i/I^{n+1}F_i \to M_i/I^{n+1}M_i \to 0.$$

Similarly we have an exact sequence

$$0 \to \operatorname{Tor}^A_{i+1}(M, A/I^n) \to M_{i+1}/I^n M_{i+1} \to F_i/I^n F_i \to M_i/I^n M_i \to 0.$$

Computing lengths we have

$$\ell\left(\frac{I^nM_i}{I^{n+1}M_i}\right)+\ell\left(\frac{I^nM_{i+1}}{I^{n+1}M_{i+1}}\right)=(\operatorname{rank}F_i)\ell\left(\frac{I^n}{I^{n+1}}\right)+\ell\left(\operatorname{Tor}^A_{i+1}(M,A/I^{n+1})\right)$$
$$-\ell\left(\operatorname{Tor}^A_{i+1}(M,A/I^{n})\right).$$

By 2.13 the terms in the right hand side are of quasi-polynomial type with period 2 and degree $\leq \operatorname{cx}(M)-1$. The result follows from Corollary 2.8. $\square$

As a consequence of the above Proposition we get:

**Corollary 2.16.** *Let $(A,\mathfrak{m})$ be an Artin local ring and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a finitely generated $A$-module of finite GCI-dimension. Then for all $j\geq 0$ the function $i\mapsto e^I_j(\operatorname{Syz}^A_i(M))$ is of quasi-polynomial type with period 2 and degree $\leq \operatorname{cx}(M)-1$.*

*Proof.* We note that $I^r=0$ for some $r\geq 1$. The result now follows from Corollary 2.15. $\square$

**2.17.** *Eisenbud operators modulo a regular element.*
Let $x$ be $A\oplus M$-regular. Let $y\in Q$ be a pre-image of $x$ in $Q$. As permutation of regular sequences is also a regular sequence we also get that $y$ is $Q$-regular. We now note that $\operatorname{projdim}_{Q/(y)}(M/xM)$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$ over $A$. Then $\mathbb{F}/x\mathbb{F}$ is a minimal resolution of $M/xM$ over $A/(x)$. Let $t_1,\dots,t_c$ be the Eisenbud operators over $\mathbb{F}$. By the construction of Eisenbud operators it follows that $t_1,\dots,t_c$ induce operators $t_1^*,\dots,t_c^*$ over $\mathbb{F}/x\mathbb{F}$. Let $N$ be an $A/(x)$-module. Set $E=\operatorname{Tor}^A_*(M,N)=\operatorname{Tor}^{A/(x)}_*(M/xM,N)$. Then the action of $t_i$ on $E$ is same as that of $t_i^*$.

**2.18.** Now assume that $Q$ be a Noetherian ring and let $\mathbf{f}=f_1,\dots f_c$ be a regular sequence in $Q$. Set $A=Q/(\mathbf{f})$. Let $M$ be a finitely generated $A$-module with $\operatorname{projdim}_Q M$ finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Let $t_1,\cdots,t_c\colon\mathbb{F}(+2)\to\mathbb{F}$ be the Eisenbud operators. Let $I$ be an $\mathfrak{m}$-primary ideal of $A$ and let $\mathcal{R}=A[Iu]$ be the Rees algebra of $A$ with respect to $I$. Let $N=\bigoplus_{n\geq 0}N_n$ be an $\mathcal{R}$-module (not necessarily finitely generated). We claim that $\bigoplus_{i\geq 0}\operatorname{Tor}^A_i(M,N)=\bigoplus_{i,n\geq 0}\operatorname{Tor}^A_i(M,N_n)$ has a bigraded $\mathcal{R}[t_1,\dots,t_c]$-module structure. Here $t_i$ have degree $(0,2)$ and if $a\in I^n$ then $\deg au^n=(n,0)$. To see this let $v=au^s\in\mathcal{R}_s$. The map $N_n\xrightarrow{v}N_{n+s}$ yields the following commutative diagram

$$\begin{tikzcd}
\mathbb{F}\otimes N_n \arrow[r, "t_j"'] \arrow[d, "v"] & \mathbb{F}\otimes N_n(-2) \arrow[d, "v"] \\
\mathbb{F}\otimes N_{n+s} \arrow[r, "t_j"'] & \mathbb{F}\otimes N_{n+s}(-2).
\end{tikzcd}$$

Taking homology we get the required result. An analogous argument yields that $\bigoplus_{i,n\geq 0}\operatorname{Ext}^i_A(M,N_n)$ is a bigraded $\mathcal{R}[t_1,\dots,t_c]$-module.

## 3. $L^I_i(M)$

In this section we extend and simplify a technique from [9] and [10]. Let $A$ be a Noetherian ring, $I$ an $\mathfrak{m}$-primary ideal and $M$ a finitely generated $A$-module.

**3.1.** Set $L_0^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. Let $\mathcal{R} = A[Iu]$ be the *Rees-algebra* of $I$. Let $\mathcal{S} = A[u]$. Then $\mathcal{R}$ is a subring of $\mathcal{S}$. Set $M[u] = M \otimes_A \mathcal{S}$ an $\mathcal{S}$-module and so an $\mathcal{R}$-module. Let $\mathcal{R}(M) = \bigoplus_{n\geq 0} I^n M$ be the Rees-module of $M$ with respect to $I$. We have the following exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}(M) \to M[u] \to L_0^I(M)(-1) \to 0.$$

Thus $L_0^I(M)(-1)$ (and so $L_0^I(M)$) is a $\mathcal{R}$-module. We note that $L_0^I(M)$ is *not* a finitely generated $\mathcal{R}$-module. Also note that $L_0^I(M) = M \otimes_A L_0^I(A)$.

**3.2.** For $i \geq 1$ set

$$L_i^I(M) = \operatorname{Tor}_i^A(M, L_0^I(A)) = \bigoplus_{n\geq 0} \operatorname{Tor}_i^A(M, A/I^{n+1}).$$

We assert that $L_i^I(M)$ is a finitely generated $\mathcal{R}$-module for $i \geq 1$. It is sufficient to prove it for $i = 1$. We tensor the exact sequence $0 \to \mathcal{R} \to \mathcal{S} \to L_0^I(A)(-1) \to 0$ with $M$ to obtain a sequence of $\mathcal{R}$-modules

$$0 \to L_1^I(M)(-1) \to \mathcal{R} \otimes_A M \to M[u] \to L_0^I(M)(-1) \to 0.$$

Thus $L_1^I(M)(-1)$ is a $\mathcal{R}$-submodule of $\mathcal{R} \otimes_A M$. The latter module is a finitely generated $\mathcal{R}$-module. It follows that $L_1^I(M)$ is a finitely generated $\mathcal{R}$-module.

**3.3.** Now assume that $A$ is Cohen-Macaulay and $M$ is maximal Cohen-Macaulay. Set $N = \operatorname{Syz}_1^A(M)$ and $F = A^{\mu(M)}$ (here $\mu(M)$ is the cardinality of a minimal generator set of $M$). We tensor the exact sequence

$$0 \to N \to F \to M \to 0,$$

with $L_0^I(A)$ to obtain an exact sequence of $\mathcal{R}$-modules

$$0 \to L_1^I(M) \to L_0^I(N) \to L_0^I(F) \to L_0^I(M) \to 0.$$

As $e_0^I(F) = e_0^I(M) + e_0^I(N)$ we get that the function $n \to \ell(\operatorname{Tor}_1^A(M, A/I^{n+1}))$ is of polynomial type and degree $\leq d-1$. Thus $\dim L_1^I(M) \leq d$.

We need the following result.

**Proposition 3.4.** *Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a MCM $A$-module. Set $M_1 = \operatorname{Syz}_1^A(M)$. Let $x$ be $A \oplus M \oplus M_1$-superficial with respect to $I$. Let $\mathcal{R} = A[Iu]$ be the Rees algebra of $A$ with respect to $I$. Set $X = xu \in \mathcal{R}_1$. If $N$ is a $\mathcal{R}$-module, let $H_1(X, N)$ denote the first Koszul homology of $N$ with respect to $X$. Then*

1. *$H_1(X, L_0^I(M)) = \bigoplus_{n\geq 0}(I^{n+1}M \colon x)/I^n M$.*
2. *$X$ is $L_0^I(M)$ regular if and only if $x^*$ is $G_I(M)$-regular.*
3. *If $x^*$ is $G_I(A)$-regular then $H_1(X, L_1^I(M)) \cong H_1(X, L_0^I(M_1))$.*

*Proof.* (1) The map $M/I^nM \xrightarrow{X} M/I^{n+1}M$ is given by $m + I^nM \mapsto xm + I^{n+1}M$. The result follows.

(2) This follows from (1).

(3) We have an exact sequence $0 \to L_1^I(M) \to L_0^I(M_1) \to L_0^I(F_0) \xrightarrow{\pi} L_0^I(M) \to 0$. Let $E = \ker \pi$. As $x^*$ is $G_I(A)$-regular we get that $H_1(X, L_0^I(A)) = 0$. So $H_1(X, E) = 0$. The result follows. $\square$

**II**: *Ratliff-Rush filtration.*

Let $(A,\mathfrak{m})$ be a Noetherian local ring and let $I$ be an $\mathfrak{m}$-primary ideal in $A$. Let $M$ be a finitely generated $A$-module. The Ratliff-Rush submodule $\widetilde{IM}$ of $M$ with respect to $I$ is defined by

$$\widetilde{IM} = \bigcup_{k\geq 0}(I^{k+1}M \colon I^k).$$

If $\operatorname{depth} M > 0$ then $\widetilde{I^kM} = I^kM$ for $k \gg 0$.

**3.5.** Now assume $d = \dim A > 0$ and $A$ is Cohen-Macaulay. Also assume $M$ is MCM $A$-module. Let $\mathcal{R}$ be the Rees algebra of $A$ with respect to $I$ and let $\mathcal{M}$ be its unique maximal graded ideal. Then by [10, 4.7] we have

$$H^0_{\mathcal{M}}(L^I_0(M)) = \bigoplus_{n\geq 0}\frac{\widetilde{I^{n+1}M}}{I^{n+1}M}.$$

We need the following result:

**Proposition 3.6.** *[with assumptions as above] If* $\operatorname{depth} G_I(A) > 0$ *then*

$$H^0_{\mathcal{M}}(L^I_1(M)) \cong H^0_{\mathcal{M}}(L^I_0(\operatorname{Syz}^A_1(M))).$$

*Proof.* Set $M_1 = \operatorname{Syz}^A_1(M)$. The exact sequence $0 \to M_1 \to F \to M \to 0$ yields an exact sequence

$$0 \to L^I_1(M) \to L^I_0(M_1) \to L^I_0(F) \to L^I_0(M) \to 0.$$

Therefore we have an exact sequence

$$0 \to H^0_{\mathcal{M}}(L^I_1(M)) \to H^0_{\mathcal{M}}(L^I_0(M_1)) \to H^0_{\mathcal{M}}(L^I_0(F))$$

Now as $\operatorname{depth} G_I(A) > 0$ we get that $\widetilde{I^n} = I^n$ for all $n \geq 1$. So $H^0_{\mathcal{M}}(L^I_0(A)) = 0$. Thus $H^0_{\mathcal{M}}(L^I_0(F)) = 0$. The result follows. $\square$

**3.7.** Now assume that $A = Q/(\mathbf{f})$ where $\mathbf{f} = f_1, \ldots, f_c$ is a $Q$-regular sequence and $M$ is a finitely generated $A$-module with $\operatorname{projdim}_Q M$ finite. Let $I$ be an $\mathfrak{m}$-primary ideal. Set $L(M) = \bigoplus_{i\geq 0} L^I_i(M) = \bigoplus_{i\geq 0}\operatorname{Tor}^A_i(M, L^I_0(A))$. Give $L(M)$ a structure of a bigraded $S = \mathcal{R}[t_1, \cdots, t_c]$-module as discussed in 2.18. Then $H^0_{\mathcal{M}}(L(M)) = \bigoplus_{i\geq 0} H^0_{\mathcal{M}}(L^I_i(M))$ is a bigraded $S$-module. To see this note that $H^0_{\mathcal{M}}(L(M)) = H^0_{\mathcal{M}S}(L(M))$.

## 4. DIMENSION ONE

In this section we assume $\dim A = 1$. We prove that if $M$ is MCM $A$-module with finite GCI dimension then the function $i \mapsto e^I_1(\operatorname{Syz}^A_i(M))$ is of quasi-polynomial type with period two. If $G_I(A)$ is Cohen-Macaulay then we show that the functions $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}^A_{2j}(M))$ and $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}^A_{2j+1}(M))$ is constant for $j \gg 0$.

The following Lemma is crucial.

**Lemma 4.1.** *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one and with an infinite residue field. Let $I$ be an $\mathfrak{m}$-primary ideal and let $x$ be $A$-superficial with respect to $I$. Assume $I^{r+1} = xI^r$. Let $M$ be a MCM $A$-module. Then*

(1) $\deg h^I(M, z) \leq r$

(2) *The postulation number of the Hilbert-Samuel function of $M$ with respect to $I$ is $\leq r - 2$.*

(3) *For $n \geq r$ and for all $i \geq 1$ we have*

$$\operatorname{Tor}_i^A(M, A/I^n) \cong \operatorname{Tor}_i^A(M, A/I^{n+1}) \quad \text{as } A\text{-modules.}$$

(4) $\widetilde{I^n M} = I^n M$ *for all $n \geq r$.*

*Proof.* We first note that as $x$ is $A$-regular it is also $M$-regular. Thus the equality $I^{r+1}M = xI^rM$ implies that $(I^{n+1}M \colon x) = I^nM$ for $n \geq r$. Therefore $x$ is $M$-superficial with respect to $I$.

(1) This follows from [8, Proposition 13].

(2) This follows from (1).

(3) Set $B = A/(x)$. For $n \geq r$ we have an exact sequence

$$0 \to A/I^n \xrightarrow{\alpha_n} A/I^{n+1} \to B \to 0.$$

Here $\alpha_n(a + I^n) = xa + I^{n+1}$. We have also used that for $n \geq r$, $I^{n+1} \subseteq (x)$. As $\operatorname{Tor}_i^A(M, B) = 0$ for $i \geq 1$ we get the required result.

(4) As $(I^{n+1}M \colon x) = I^nM$ for all $n \geq r$, by [10, 2.7] we get $\widetilde{I^nM} = I^nM$ for $n \geq r$. □

As an immediate corollary we obtain

**Theorem 4.2.** *Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension $d$ and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a MCM $A$-module. Assume $M$ has finite GCI-dimension over $A$. Then the function $i \mapsto e_1^I(\operatorname{Syz}_i^A(M))$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$.*

*Proof.* By Corollary 2.16 the result holds when $A$ is Artin. By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}_i^A(M)$. If $d \geq 2$ then by 2.2 we may choose $\mathbf{x} = x_1, \ldots, x_{d-1}$ which is $A \oplus M_i$-superficial for all $i \geq 0$. As $e_1^I(M_i) = e_1^I(M_i/\mathbf{x}M_i)$ for all $i \geq 0$, we may assume $\dim A = 1$.

Let $x$ be $A$-superficial with respect to $I$. Say $I^{r+1} = xI^r$. Then by Lemma 4.1 $x$ is $M_i$-superficial for all $i \geq 0$ and the postulation number of the Hilbert-Samuel function of $M_i$ with respect to $I$ is $\leq r - 2$.

Fix $n \geq r$. The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ yields an exact sequence

$$0 \to \operatorname{Tor}_{i+1}^A(M, A/I^{n+1}) \to M_{i+1}/I^{n+1}M_{i+1} \to F_i/I^{n+1}F_i \to M_i/I^{n+1}M_i \to 0.$$

Using Lemma 4.1, it follows that

$$\ell(\operatorname{Tor}_{i+1}^A(M, A/I^{n+1})) = (\operatorname{rank} F_i)e_1^I(A) - e_1^I(M_i) - e_1^I(M_{i+1}).$$

The functions $i \mapsto \operatorname{rank} F_i$ and $i \mapsto \ell(\operatorname{Tor}_i^A(M, A/I^n))$ are of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$. The result now follows from 2.8. □

We now prove:

**Theorem 4.3.** *Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension one and let $I$ be an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Let $M$ be a MCM $A$-module. Assume $M$ has finite GCI dimension over $A$. Then the functions $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}_{2j}^A(M))$ and $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}_{2j+1}^A(M))$ are constant for $j \gg 0$*

*Proof.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with infinite residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}_i^A(M)$.

Let $x$ be $I$-superficial. Then $x^*$ is $G_I(A)$-regular. Say $I^{r+1} = xI^r$. Then by Lemma 4.1 we get that $x$ is $M_i$-superficial for all $i \geq 0$. We have $(I^{n+1}M_i \colon x) = I^n M_i$ for all $n \geq r$ and for all $i \geq 0$.

Let $t_1, \ldots, t_c$ be the Eisenbud operators on $\mathbb{F}$. Let $\mathcal{R}$ be the Rees algebra of $A$ with respect to $I$. By 2.18 we get that $E = \bigoplus_{i \geq 0} L_i^I(M)$ is a bigraded $S = \mathcal{R}[t_1, \ldots, t_c]$-module. Set $X = xu$. Then $H_1(X, E)$ is a bigraded $S$-module. By 3.4 we get that $H_1(X, E) = \bigoplus_{i \geq 0} H_1(X, L_0^I(M_i))$. We now note that $H_1(X, E) \subseteq E_{n \leq r-1}$ and so is an Artinian $A[t_1, \ldots, t_c]$-module. It follows that the function $j \mapsto \ell(H_1(X, L_0^I(M_i)))$ is of quasi-polynomial type with period two. By 3.4 the result follows. □

We now prove

**Proposition 4.4.** *Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension one and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a MCM $A$-module of finite GCI dimension over $A$. Then the function $i \mapsto e_2^I(\operatorname{Syz}_i^A(M))$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$.*

*Proof.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with infinite residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}_i^A(M)$.

Let $x$ be $I$-superficial. Say $I^{r+1} = xI^r$. Then by Lemma 4.1 we get that

$$\operatorname{Tor}_i^A(M, A/I^n) \cong \operatorname{Tor}_i^A(M, A/I^{n+1}) \quad \text{for all } n \geq r. \tag{1}$$

Fix $n \geq 0$. Then by 2.13 the function $i \mapsto \ell(\operatorname{Tor}_i^A(M, A/I^{n+1}))$ is quasi-polynomial of period two and degree $\leq \operatorname{cx}(M) - 1$. Thus by (1) we get that there exists $p(z, w) \in \mathbb{Z}[z, w]$ such that

$$\sum_{i \geq 1, n \geq 0} \ell(\operatorname{Tor}_i^A(M, A/I^{n+1})) z^i w^n = \frac{p(z, w)}{(1 - z^2)^{\operatorname{cx}(M)}(1 - w)}.$$

Set

$$f(i, n) = \sum_{m=0}^{n} \ell(\operatorname{Tor}_i^A(M, A/I^{m+1})).$$

Multiplying the previous equation by $1/(1 - w)$ we get that

$$\sum_{i \geq 1, n \geq 0} f(i, n) z^i w^n = \frac{p(z, w)}{(1 - z^2)^{\operatorname{cx}(M)}(1 - w)^2}.$$

By 2.9 we get that for $i, n \gg 0$

$$f(i, n) = g_0(i)(n + 1) - g_1(i)$$

where for $j = 0, 1$ the function $i \mapsto g_j(i)$ is of quasi-polynomial type with period 2 and degree $\leq \operatorname{cx}(M) - 1$. It remains to note that

$$g_0(i) = (\operatorname{rank} F_i) e_1^I(A) - e_1^I(M_i) - e_1^I(M_{i-1}), \text{and}$$
$$g_1(i) = (\operatorname{rank} F_i) e_2^I(A) - e_2^I(M_i) - e_2^I(M_{i-1}).$$

The result now follows from Corollary 2.8. □

We need the following result in the next section.

**Proposition 4.5.** *Let $(A, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring with infinite residue field and let $I$ be an $\mathfrak{m}$-primary ideal of $A$. Let $M$ be a maximal Cohen-Macaulay $A$-module. Assume $A = Q/(\mathbf{f})$ where $\mathbf{f} = f_1, \ldots, f_c$ is a $Q$-regular sequence and* $\operatorname{projdim}_Q M$ *is finite. Let $\mathbb{F}$ be a minimal resolution of $M$ over $A$. Let $t_1, \ldots, t_c$ be the Eisenbud operators over $\mathbb{F}$. Let $\mathcal{R} = A[Iu]$ be the Rees algebra of $A$ with respect to $I$ and let $\mathcal{M}$ be its unique maximal homogeneous ideal. Then $\bigoplus_{i\geq 0} H^0_{\mathcal{M}}(L^I_i(M))$ is an Artinian $A[t_1, \ldots, t_c]$-module. Furthermore the function $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_i(M)))$ is of quasi-polynomial type with period two and degree* $\leq \operatorname{cx}(M) - 1$.

*Proof.* Let $x$ be $A$-superficial with respect to $I$. Say $I^{r+1} = xI^r$. Set $M_i = \operatorname{Syz}^A_i(M)$ for $i \geq 0$. Then by Lemma 4.1.4 we get that $\widetilde{I^n M_i} = I^n M_i$ for $n \geq r$. So $H^0_{\mathcal{M}}(L^I_0(M_i))_n = 0$ for $n \geq r - 1$. The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ yields an exact sequence

$$0 \to L^I_{i+1}(M) \to L^I_0(M_{i+1}) \to L^I_0(F_i) \to L^I_0(M_i) \to 0.$$

So we have an inclusion $0 \to H^0_{\mathcal{M}}(L^I_{i+1}(M)) \to H^0_{\mathcal{M}}(L^I_0(M_{i+1}))$. It follows that $H^0_{\mathcal{M}}(L^I_{i+1}(M))_n = 0$ for $n \geq r - 1$.

By 2.18, $L = \bigoplus_{i\geq 0} L^I_i(M)$ is a bigraded $S = \mathcal{R}[t_1, \ldots, t_c]$-module. Then $H^0_{\mathcal{M}}(L) = \bigoplus_{i\geq 0} H^0_{\mathcal{M}}(L^I_i(M))$ is also a bigraded $S$-module. As $H^0_{\mathcal{M}}(L) \subseteq E = L_{n\leq r-1}$ it follows that $H^0_{\mathcal{M}}(L)$ is an Artinian $A[t_1, \ldots, t_c]$-module. It follows that the function $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_i(M)))$ is of quasi-polynomial type with period two. To compute its degree note that the function $i \mapsto \ell(E_i)$ has degree $\leq \operatorname{cx}(M) - 1$.

□

## 5. DIMENSION 2

In this section we assume $(A, \mathfrak{m})$ is Cohen-Macaulay with $\dim A = 2$ . Let $I$ be an $\mathfrak{m}$-primary ideal. If $M$ is a MCM $A$-module with finite GCI dimension over $A$ then we prove that the function $i \mapsto e^I_2(\operatorname{Syz}^A_i(M))$ is of quasi-polynomial type with period two. If $G_I(A)$ is Cohen-Macaulay then we show that the functions $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}^A_{2j}(M))$ and $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}^A_{2j+1}(M))$ are constant for $j \gg 0$.

**Theorem 5.1.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$ and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a MCM $A$-module. Assume that $M$ has finite GCI dimension over $A$. Then the function $i \mapsto e^I_2(\operatorname{Syz}^A_i(M))$ is of quasi-polynomial type with period two and degree* $\leq \operatorname{cx}(M) - 1$.

*Proof.* By Corollary 2.16 the result holds when $A$ is Artin. By Proposition 4.4 the result holds when $d = 1$. By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}^A_i(M)$. If $d \geq 3$ then by 2.2 we may choose $\mathbf{x} = x_1, \ldots, x_{d-2}$ which is $A \oplus M_i$-superficial for all $i \geq 0$. As $e^I_2(M_i) = e^I_2(M_i/\mathbf{x}M_i)$ for all $i \geq 0$, we may assume $\dim A = 2$.

We note that for any $n \geq 1$ we have $e^{I^n}_2(M) = e^I_2(M)$. For $n \gg 1$ we have $\operatorname{depth} G_{I^n}(A) \geq 1$. Thus we can assume $\operatorname{depth} G_I(A) > 0$. Let $x$ be $A \oplus M_i$-superficial for all $i \geq 0$. Then $x^*$ is $G_I(A)$-regular. Let $\mathcal{R} = A[Iu]$ be the Rees algebra of $A$ with respect to $I$. Set $X = xu \in \mathcal{R}_1$ and $\overline{A} = A/(x)$. Also set $\overline{M_i} = M_i/xM_i$.

Let $t_1, \ldots, t_c$ be the Eisenbud operators over $\mathbb{F}$. Then $L(M) = \bigoplus_{i \geq 0} L_i^I(M)$ is a bigraded $S = \mathcal{R}[t_1, \ldots, t_c]$-module (see 3.7). Notice $X$ is $L_0^I(A)$-regular. So we have an exact sequence of $\mathcal{R}$-modules

$$0 \to L_0^I(A)(-1) \xrightarrow{X} L_0^I(A) \to L_0^I(\overline{A}) \to 0.$$

This induces an exact sequence of $S$-modules

$$0 \to K \to L(M)(-1, 0) \xrightarrow{X} L(M) \to L(\overline{M}) \tag{2}$$

By Proposition 3.4 we get that

$$K = \bigoplus_{i \geq 0} H_1(X, L_0^I(M_i)). \tag{3}$$

Let $\mathcal{M}$ be the unique maximal homogeneous maximal ideal of $\mathcal{R}$. We take local cohomology with respect to $\mathcal{M}$ (on (2)). As $K$ is $\mathcal{M}$-torsion we get an exact sequence of $S$-modules

$$0 \to K \to H^0_{\mathcal{M}}(L(M))(-1, 0) \to H^0_{\mathcal{M}}(L(M)) \xrightarrow{\rho} H^0_{\mathcal{M}}(L(\overline{M})). \tag{4}$$

By 2.17 and 4.5 we get that $H^0_{\mathcal{M}}(L(\overline{M}))$ is an Artin module over the subring $T = A[t_1, \cdots, t_c]$ of $S$. So $E = \operatorname{image} \rho$ is also an Artin $T$-module. By 3.6 we get that

$$H^0_{\mathcal{M}}(L(M)) = \bigoplus_{i \geq 0} H^0_{\mathcal{M}}(L_0^I(M_i)).$$

Thus for a fixed $i$ we get that $H^0_{\mathcal{M}}(L(M))_{(i,n)} = 0$ for $n \gg 0$. It follows that $\ell(K_i) = \ell(E_i)$. As $E$ is an Artin $T$-module the function $i \mapsto \ell(E_i)$ is of quasi-polynomial type with period two. By 4.5 its degree is $\leq \operatorname{cx}(M) - 1$. Thus the function $i \mapsto \ell(K_i)$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$. By 2.3(3b) 3.4(1) and (3) we have

$$e_2^I(M_i) = e_2^I(\overline{M_i}) - \ell(K_i),$$

is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$. $\square$

Next we prove:

**Theorem 5.2.** *Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension two and let $I$ be an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Let $M$ be a MCM $A$-module. Assume $M$ has finite GCI dimension over $A$. Then the functions $j \mapsto$ $\operatorname{depth} G_I(\operatorname{Syz}^A_{2j}(M))$ and $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}^A_{2j+1}(M))$ are constant for $j \gg 0$*

To prove this theorem we need the following:

**Lemma 5.3.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be an MCM $A$-module. Let $x, y$ be an $A \oplus M$-superficial sequence. Set $J = (x, y)$ and assume $I^{r+1} = JI^r$. Then the following are equivalent:*

(1) *$x^*$ is $G_I(M)$-regular.*
(2) *$(I^{n+1}M \colon x) = I^n M$ for all $n \geq 1$.*
(3) *$(I^{n+1}M \colon x) = I^n M$ for $n = 1, \cdots, r$.*

*Proof.* The equivalence (1) and (2) is well-known. Also clearly (2) implies (3). Now assume (3). Set $N = M/xM$. We have the following exact sequence

$$0 \to \frac{(I^n M \colon J)}{I^{n-1}M} \xrightarrow{\gamma_n} \frac{(I^n M \colon x)}{I^{n-1}M} \xrightarrow{\beta_n} \frac{(I^{n+1} M \colon x)}{I^n M} \xrightarrow{\alpha_n} \frac{I^{n+1}M}{JI^n M} \xrightarrow{\rho_n} \frac{I^{n+1}N}{yI^n N} \to 0,$$

where $\rho_n$ is the natural surjection, $\gamma_n$ is the natural inclusion, $\alpha_n(a + I^n M) = xa + JI^n M$ and $\beta_n(a + I^{n-1}M) = ya + I^n M$. Set

$$U = \bigoplus_{n \geq 0} \frac{(I^n M \colon J)}{I^{n-1}M} \quad \text{and} \quad V = \bigoplus_{n \geq 0} \frac{(I^{n+1}M \colon x)}{I^n M}.$$

If $(I^{n+1}M \colon x) = I^n M$ for $n = 1, \cdots, r$ then by the above exact sequence we have an exact sequence

$$0 \to U \to V[-1] \to V \to 0$$

As $\ell(V)$ is finite we get that $V[-1] \cong V$ and this implies $V = 0$, so (2) holds. □

We now give

*Proof of Theorem 5.2.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \operatorname{Syz}_i^A(M)$.

Let $x, y$ be $M_i \oplus A$-superficial sequence with respect to $I$ for all $i$. Such an element exists as the residue field of $A$ is uncountable, see 2.2. Then $x^*$ is $G_I(A)$-regular. Let $\mathcal{R} = A[Iu]$ be the Rees algebra of $A$ with respect to $I$. Set $X = xu \in \mathcal{R}_1$ and $\overline{A} = A/(x)$. Also set $\overline{M_i} = M_i/xM_i$.

Let $t_1, \ldots, t_c$ be the Eisenbud operators over $\mathbb{F}$. Then $L(M) = \bigoplus_{i \geq 0} L_i^I(M)$ is a bigraded $S = \mathcal{R}[t_1, \ldots, t_c]$-module (see 3.7).

Notice $X$ is $L_0^I(A)$-regular. So we have an exact sequence of $\mathcal{R}$-modules

$$0 \to L_0^I(A)(-1) \xrightarrow{X} L_0^I(A) \to L_0^I(\overline{A}) \to 0.$$

This induces an exact sequence of $S$-modules

$$0 \to K \to L(M)(-1, 0) \xrightarrow{X} L(M) \to L(\overline{M}). \tag{5}$$

By Proposition 3.4 we get that

$$K = \bigoplus_{i \geq 0} H_1(X, L_0^I(M_i)). \tag{6}$$

Let $I^{r+1} = (x, y)I^r$. We note that $E = K_{n \leq r}$ is an Artin $A[t_1, \ldots, t_c]$-module. Thus the function $i \to \ell(E_i)$ is of quasi-polynomial type with period two. Therefore by Lemma 5.3 we get that for each $j = 0, 1$ either $\operatorname{depth} G_I(M_{2i+j}) = 0$ for $i \gg 0$ or $\operatorname{depth} G_I(M_{2i+j}) \geq 1$ for $i \gg 0$. We now go mod $x$ and use Theorem 4.3 and 2.3(5) to conclude. □

## 6. ASYMPTOTIC DEPTH

In this section $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension two and $I$ is an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Let $M$ be a MCM $A$-module with finite GCI-dimension over $A$. Let $\xi_I(M) = \lim_{n \to \infty} \operatorname{depth} G_{I^n}(M)$. In this section we prove that the functions $i \mapsto \xi_I(\operatorname{Syz}_{2i}^A(M))$ and $i \mapsto \xi_I(\operatorname{Syz}_{2i+1}^A(M))$ are constant for $i \gg 0$. Let $\mathcal{R}(I)$ denote the Rees algebra of $A$ with respect to $I$ and let $\mathcal{M}$ denote its unique maximal homogeneous ideal.

**6.1.** We note that $\xi_I(M) \geq 1$. Furthermore $\xi_I(M) \geq 2$ if and only if $H^1_{\mathcal{M}}(L_0^I(M))_{-1} = 0$, see [10, 9.2]. As $\dim A = 2$ the only possible values of $\xi_I(M)$

is 1 or 2. We now note that $L_0^I(M)(-1) = \bigoplus_{n\geq 0} M/I^nM$ behaves very well with respect to to the Veronese functor. Clearly for $m \geq 1$ we have

$$L_0^I(M)(-1)^{<m>} = \left(\bigoplus_{n\geq 0} M/I^nM\right)^{<m>} = \bigoplus_{n\geq 0} M/I^{nm} = L_0^{I^m}(M)(-1).$$

Also note that $\mathcal{M}^{<m>}$ is the unique maximal homogeneous ideal of $\mathcal{R}(I^m)$. It follows that

$$H^1_{\mathcal{M}}(L_0^I(M))_{-1} \cong H^1_{\mathcal{M}^{<m>}}(L_0^{I^m}(M))_{-1} \quad \text{as } A\text{-modules.}$$

We need the following:

**Lemma 6.2.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two and let $I$ be an $\mathfrak{m}$-primary ideal. Let $J = (x, y)$ be a minimal reduction of $I$ and assume that $I^{r+1} = JI^r$. Let $M$ be a maximal Cohen-Macaulay $A$-module. Then $H^1_{\mathcal{M}}(L_0^I(M))_n = 0$ for $n \geq r-2$. In particular if $m \geq r$ then*

$$H^1_{\mathcal{M}^{<m>}}(L_0^{I^m}(M))_n = 0 \quad \textit{for } n \geq 0.$$

*Proof.* Let $a_2(G_I(M)) = \max\{n \mid H^2_{\mathcal{M}}(G_I(M))_n \neq 0\}$. Then by [13, 3.2] we get that $a_2(G_I(M)) \leq r-2$. The exact sequence $0 \to G_I(M) \to L_0^I(M) \to L_0^I(M)(-1) \to 0$ induces an exact sequence

$$H^1_{\mathcal{M}}(L_0^I(M)) \to H^1_{\mathcal{M}}(L_0^I(M))(-1) \to H^2_{\mathcal{M}}(G_I(M)).$$

By [10, 6.4] we get that $H^1_{\mathcal{M}}(L_0^I(M))_n = 0$ for $n \gg 0$. From the above exact sequence it follows that $H^1_{\mathcal{M}}(L_0^I(M))_n = 0$ for all $n \geq r-2$. The rest of the assertion follows from 6.1. $\square$

We now state and prove the main result of this section.

**Theorem 6.3.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two and let $I$ be an $\mathfrak{m}$-primary ideal with $G_I(A)$ Cohen-Macaulay. Let $M$ be a MCM $A$-module and assume $M$ has finite GCI dimension over $A$. Then the functions $i \mapsto \xi_I(\mathrm{Syz}^A_{2i}(M))$ and $i \mapsto \xi_I(\mathrm{Syz}^A_{2i+1}(M))$ are constant for $i \gg 0$.*

*Proof.* By Remark 2.5 we may assume $A = Q/(\mathbf{f})$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Set $M_i = \mathrm{Syz}^A_i(M)$. Let $\mathcal{R}(I) = A[Iu]$ denote the Rees algebra of $A$ with respect to $I$ and let $\mathcal{M}$ denote its unique maximal homogeneous ideal. By Lemma 6.2 and 6.1 we may assume that $H^1_{\mathcal{M}}(L_0^I(M_i))_n = 0$ for $n \geq 0$ and for all $i \geq 0$. We will show that the vanishing of $H^1_{\mathcal{M}}(L_0^I(M_i))_{-1}$ can be detected by a quasi-polynomial of period two.

*Claim-1:* $H^1_{\mathcal{M}}(L_1^I(M_i))_{-1} \cong H^1_{\mathcal{M}}(L_0^I(M_{i+1}))_{-1}$ and $H^1_{\mathcal{M}}(L_1^I(M_i))_j = 0$ for $j \geq 0$.

The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ induces an exact sequence $0 \to L_1^I(M_i) \to L_0^I(M_{i+1}) \to L_0^I(F_i) \xrightarrow{\pi_i} L_0^I(M_i) \to 0$. Let $C = \ker \pi_i$.

As $G_I(A)$ is Cohen-Macaulay we have $H^i_{\mathcal{M}}(L_0^I(A)) = 0$ for $i = 0, 1$, see [10, 5.2]. Thus $H^0_{\mathcal{M}}(C) = 0$ and $H^1_{\mathcal{M}}(C) \cong H^0_{\mathcal{M}}(L_0^I(M_i))$. In particular we have $H^1_{\mathcal{M}}(C)_{-1} = 0$.

The exact sequence $0 \to L_1^I(M_i) \to L_0^I(M_{i+1}) \to C \to 0$ induces an exact sequence

$$0 \to H^1_{\mathcal{M}}(L_1^I(M_i)) \to H^1_{\mathcal{M}}(L_0^I(M_{i+1})) \to H^1_{\mathcal{M}}(C)$$

As $H^1_{\mathcal{M}}(C)_{-1} = 0$ we get $H^1_{\mathcal{M}}(L^I_1(M_i))_{-1} \cong H^1_{\mathcal{M}}(L^I_0(M_{i+1}))_{-1}$. As $H^1_{\mathcal{M}}(L^I_0(M_{i+1}))_j = 0$ for $j \geq 0$ we also get $H^1_{\mathcal{M}}(L^I_1(M_i))_j = 0$ for $j \geq 0$. Thus the Claim is proved.

Let $x$ be $M_i \oplus A$-superficial for each $i \geq 0$. Set $X = xu$, $\overline{A} = A/(x)$ and $\overline{M_i} = M_i/xM_i$. As $x^*$ is $G_I(A)$-regular, we have an exact sequence

$$0 \to L^I_0(A)(-1) \xrightarrow{X} L^I_0(A) \to L^I_0(\overline{A}) \to 0.$$

After tensoring with $M$ this induces for $i \geq 0$ an exact sequence

$$L^I_{i+1}(M)(-1) \xrightarrow{X} L^I_{i+1}(M) \to L^I_{i+1}(\overline{M}) \to L^I_i(M)(-1) \xrightarrow{X} L^I_i(M).$$

We note that $L^I_{i+1}(M) \cong L^I_1(M_i)$. Also by 3.4 we get that $H_1(X, L^I_1(M_i)) \cong H_1(X, L^I_0(M_{i+1}))$. Thus we have an exact sequence

$$\begin{aligned} 0 \to H_1(X, L^I_0(M_{i+1})) \to L^I_1(M_i)(-1) \xrightarrow{X} L^I_1(M_i) \to \\ \xrightarrow{\alpha_i} L^I_1(\overline{M_i}) \to H_1(X, L^I_0(M_i)) \to 0. \end{aligned}$$

Let $W_i = \ker \alpha_i$ and $D_i = \operatorname{image} \alpha_i$. As $H_1(X, L^I_0(M_{i+1}))$ has finite length we get a surjection $H^0_{\mathcal{M}}(L^I_1(M_i))(-1) \to H^0_{\mathcal{M}}(W_i)$ and a isomorphism $H^1_{\mathcal{M}}(W_i) \cong H^1_{\mathcal{M}}(L^I_1(M_i))(-1)$. Thus

$$H^0_{\mathcal{M}}(W_i)_0 = 0 \quad \text{and} \quad H^1_{\mathcal{M}}(W_i)_0 \cong H^1_{\mathcal{M}}(L^I_1(M_i))_{-1}. \tag{7}$$

The exact sequence $0 \to W_i \to L^I_1(M_i) \to D_i \to 0$ induces an exact sequence

$$0 \to H^0_{\mathcal{M}}(W_i) \to H^0_{\mathcal{M}}(L^I_1(M_i)) \to H^0_{\mathcal{M}}(D_i) \to H^1_{\mathcal{M}}(W_i) \to H^1_{\mathcal{M}}(L^I_1(M_i))$$

Evaluating at $n = 0$ we get an exact sequence

$$0 \to H^0_{\mathcal{M}}(L^I_1(M_i))_0 \to H^0_{\mathcal{M}}(D_i)_0 \to H^1_{\mathcal{M}}(W_i)_0 \to 0. \tag{8}$$

By the exact sequence $0 \to D_i \to L^I_1(\overline{M_i}) \to H_1(X, L^I_0(M_i)) \to 0$ we obtain an exact sequence

$$0 \to H^0_{\mathcal{M}}(D_i) \to H^0_{\mathcal{M}}(L^I_1(\overline{M_i})) \to H_1(X, L^I_0(M_i))$$

As $H_1(X, L^I_0(M_i))_0 = 0$ we get $H^0_{\mathcal{M}}(D_i)_0 \cong H^0_{\mathcal{M}}(L^I_1(\overline{M_i}))_0$. By (8) we get that

$$\ell(H^0_{\mathcal{M}}(L^I_1(M_i))_0) \leq \ell(H^0_{\mathcal{M}}(L^I_1(\overline{M_i}))_0) \quad \text{with equality iff } H^1_{\mathcal{M}}(W_i)_0 = 0.$$

The latter condition holds by (7) if and only if $H^1_{\mathcal{M}}(L^I_1(M_i))_{-1} = 0$ and this by our claim holds if and only if $H^1_{\mathcal{M}}(L^I_0(M_{i+1}))_{-1} = 0$. This holds if and only if $\xi_I(M_{i+1}) \geq 2$.

We now claim that the functions $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_1(M_i))_0)$ and $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_1(\overline{M_i}))_0)$ are of quasi-polynomial type with period two. This will prove our assertion. To see that these functions are of quasi-polynomial type, let $t_1, \ldots, t_c$ be the Eisenbud operators over $\mathbb{F}$. Then $L(M) = \bigoplus_{i\geq 0} L^I_i(M)$ is a bigraded module over $S = \mathcal{R}[t_1, \ldots, t_c]$, see 3.7. Thus $H^0_{\mathcal{M}}(L(M)) = \bigoplus_{i \geq 0} H^0_{\mathcal{M}}(L^I_i(M))$ is a bigraded module over $S$. By 3.6 we get that $H^0_{\mathcal{M}}(L^I_i(M))_0 \cong \widetilde{IM_i}/IM_i$. As $\bigoplus_{i\geq 0} L^I_i(M)_0$ is an Artin $A[t_1, \ldots, t_c]$-module we get that $\bigoplus_{n\geq 0} H^0_{\mathcal{M}}(L^I_i(M))_0$ is an Artin $A[t_1, \ldots, t_c]$-module. It follows that the function $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_1(M_i))_0)$ is of quasi-polynomial type with period two. A similar argument yields that the function $i \mapsto \ell(H^0_{\mathcal{M}}(L^I_1(\overline{M_i}))_0)$ is of quasi-polynomial type with period two. □

## 7. Dual Hilbert Coefficients

In this section we assume that $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring with a canonical module $\omega$. Let $I$ be an $\mathfrak{m}$-primary ideal and let $M$ be a maximal Cohen-Macaulay $A$-module. The function $D^I(M,n) = \ell(\operatorname{Hom}_A(M, \omega/I^{n+1}\omega)$ is called the *dual Hilbert-Samuel* function of $M$ with respect to $I$. In [12] it is shown that there exist a polynomial $t^I(M,z) \in \mathbb{Q}[z]$ of degree $d$ such that $t^I(M,n) = D^I(M,n)$ for all $n \gg 0$. We write

$$t^I(M,X) = \sum_{i=0}^{d} (-1)^i c_i^I(M) \binom{X+r-i}{r-i}.$$

The integers $c_i^I(M)$ are called the $i^{th}$- dual Hilbert coefficient of $M$ with respect to $I$. The zeroth dual Hilbert coefficient $c_0^I(M)$ is equal to $e_0^I(M)$. We prove:

**Theorem 7.1.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with a canonical module $\omega$ and let $I$ be an $\mathfrak{m}$-primary ideal. Let $M$ be a maximal Cohen-Macaulay A-module. Assume $M$ has finite GCI dimension. Then for $i = 0, 1, \cdots, d$ the function $j \mapsto c_i^I(\operatorname{Syz}_j^A(M))$ is of quasi-polynomial type with period two. If $A$ is a complete intersection then the degree of each of the above functions $\leq \operatorname{cx}(M) - 1$.*

*Proof.* By Remark 2.5 we may assume $A = Q/(f_1, \ldots, f_c)$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. Let $\mathbb{F}$ be a minimal resolution of $M$. Let $t_1, \ldots, t_c$ be the Eisenbud operators over $\mathbb{F}$. Let $\mathcal{R} = A[ut]$ be the Rees algebra of $A$ with respect to $I$ and let $N = \bigoplus_{n \geq 0} N_n$ is a finitely generated $\mathcal{R}$-module. Give $E(N) = \bigoplus_{i,n \geq 0} \operatorname{Ext}_A^i(M, N_n)$ a bi-graded $S = \mathcal{R}[t_1, \ldots, t_c]$ structure as described in 2.18. Then by [11, 1.1], $E(N)$ is a finitely generated $S$-module. In particular $E_I(\omega, M) = \bigoplus_{i,n \geq 0} \operatorname{Ext}_A^i(M, I^n\omega)$ is a finitely generated $S$-module. We now note that
$\operatorname{Ext}_A^i(M, \omega/I^{n+1}\omega) \cong \operatorname{Ext}_A^{i+1}(M, I^{n+1}\omega)$ for all $i \geq 1$ and for all $n \geq 0$. Thus

$$D_I(\omega, M) = \bigoplus_{i \geq 1, n \geq 0} \operatorname{Ext}_A^i(M, \omega/I^{n+1}\omega)$$

is a submodule of $E_I(\omega, M)(0,1)$ and so is finitely generated. Set
$K = \operatorname{ann}_S D_I(\omega, M)$. Then $\mathfrak{q} = K_{0,0}$ is $\mathfrak{m}$-primary and so in particular $D_I(\omega, M)$ is a finitely generated $T = \mathcal{R}/\mathfrak{q}\mathcal{R}[t_1, \ldots, t_c]$-module. Therefore the Hilbert series of $D_I(\omega, M)$ is of the form

$$\frac{h(z,w)}{(1-z)^d(1-w^2)^c}$$

By 2.9 we get that for $i, n \gg 0$

$$\ell(\operatorname{Ext}_A^i(M, \omega/I^{n+1}\omega)) = \sum_{l=0}^{d-1} (-1)^l v_l^I(i) \binom{n+d-1-l}{d-1-l}$$

where the functions $i \mapsto v_l^I(i)$ for $i = 0, \ldots, d-1$ are of quasi-polynomial type with period two and degree $\leq c-1$. We note that it is possible that some of the $v_l^I(i)$ is identically zero.
Now set $M_i = \operatorname{Syz}_i^A(M)$. The exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$ induces

an exact sequence

$$
\begin{aligned}
0 \to &\operatorname{Hom}_A(M_i, \omega/I^{n+1}\omega) \to \operatorname{Hom}_A(F_i, \omega/I^{n+1}\omega) \to \operatorname{Hom}_A(M_{i+1}, \omega/I^{n+1}\omega) \\
\to &\operatorname{Ext}^{i+1}_A(M, \omega/I^{n+1}\omega) \to 0.
\end{aligned}
$$

Thus for $1 \leq l \leq d$ we have that

$$\operatorname{rank}(F_i)e^I_l(\omega) - c^I_l(M_i) - c^I_l(M_{i+1}) = v^I_{l-1}(i+1).$$

Using 2.8 we get that for $1 \leq l \leq d$ the function $i \mapsto c^I_l(M_i)$ is of quasi-polynomial type with period two and degree $\leq c-1$. Also note that $c^I_0(M_i) = e^I_0(M_i)$. So for $l = 0$ the result follows from 2.14.

If $A$ is a complete intersection then we may take $Q$ to be a complete regular local ring. Then by [1, 3.9] there exists a complete local ring $R$ with $A = R/(g_1, \ldots, g_r)$ with $r = \operatorname{cx}(M)$, $g_1, \ldots, g_r$ a $R$-regular sequence and $\operatorname{projdim}_R M$ finite. It now follows that the degree of the functions $i \mapsto c^I_l(M_i)$ is $\leq \operatorname{cx}(M) - 1$. □

## 8. REGULARITY

Let $H^i(-)$ denote the $i^{th}$ local cohomology functor of $G_I(A)$ with respect to $G_I(A)_+ = \bigoplus_{n>0} I^n/I^{n+1}$. Set

$$a_i(G_I(M)) = \max\{j \mid H^i(G_I(M))_j \neq 0\}.$$

Assume that the residue field of $A$ is infinite. Let $J$ be a minimal reduction of $I$. Say $I^{r+1} = JI^r$. Let $M$ be a maximal Cohen-Macaulay $A$-module. Then it is well-known that $a_d(G_I(M)) \leq r - d$. We prove

**Theorem 8.1.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ and let $M$ be a maximal Cohen-Macaulay $A$-module. Let $I$ be an $\mathfrak{m}$-primary ideal. Assume $M$ has finite GCI dimension. Then the set*

$$\left\{ \frac{a_{d-1}(G_I(\operatorname{Syz}^A_i(M))}{i^{\operatorname{cx}(M)-1}} \right\}_{i\geq 1}$$

*is bounded.*

*Proof.* By Remark 2.5 we may assume $A = Q/(f_1, \ldots, f_c)$ where $Q$ is complete with uncountable residue field, $\mathbf{f} = f_1, \cdots, f_c$ is a $Q$-regular sequence and $\operatorname{projdim}_Q M$ is finite. We prove the result by induction on $d = \dim A$.

We first consider the case when $d = 2$. Set $M_i = \operatorname{Syz}^A_i(M)$ for $i \geq 0$. As $k$ is uncountable we can choose $x, y$ an $M_i \oplus A$-superficial sequence for all $i \geq 0$. Set $J = (x, y)$. Then $I^{r+1} = JI^r$ for some $r \geq 0$. Let $\mathcal{R} = A[ut]$ be the Rees algebra of $A$ with respect to $I$ and let $\mathcal{M}$ be its unique maximal homogeneous ideal. We note that for all $i \geq 0$ we have an isomorphism of $\mathcal{R}$-modules $H^i_{\mathcal{M}}(G_I(E)) \cong H^i_{G_I(A)_+}(G_I(E))$ for any finitely generated $A$-module $E$. The exact sequence of $\mathcal{R}$-modules $0 \to G_I(M_i) \to L^I_0(M_i) \to L^I_0(M_i)(-1) \to 0$ induces an exact sequence for all $n \in \mathbb{Z}$

$$
\begin{aligned}
H^0_{\mathcal{M}}(L^I_0(M_i))_{n-1} \to H^1(G_I(M_i))_n \to H^1_{\mathcal{M}}(L^I_0(M_i))_n \to H^1_{\mathcal{M}}(L^I_0(M_i))_{n-1} \to \\
\to H^2(G_I(M_i))_n.
\end{aligned}
$$

As $H^2(G_I(M_i))_n = 0$ for $n \geq r-1$ we get that $H^I_{\mathcal{M}}(L^I_0(M_i))_n = 0$ for $n \geq r-2$. Set $\rho^I(M_i) = \min\{j \mid \widetilde{I^n M_i} = I^n M_i \text{ for all } n \geq j\}$. As $H^0_{\mathcal{M}}(L^I_0(M_i)) = \bigoplus_{n\geq 0} \widetilde{I^{n+1}M}/I^{n+1}M$, to prove our result it suffices to show that

$$\left\{\frac{\rho^I(M_i)}{i^{\operatorname{cx}(M)-1}}\right\}_{i\geq 1} \quad \text{is bounded.}$$

We have nothing to show if $\operatorname{cx}(M) \leq 1$ for then either $M$ is free (if $\operatorname{cx}(M) = 0$) or $M$ has a periodic resolution with period two (if $\operatorname{cx}(M) = 1$). So assume $\operatorname{cx}(M) \geq 2$.

Set

$$B^I(M_i) = \bigoplus_{n\geq 0} \frac{I^{n+1}M_i \colon x}{I^n M_i} = H_1(xu, L^I_0(M_i)).$$

Set $\overline{M_i} = M_i/xM_i$. By [10, 2.9] we have an exact sequence

$$0 \to \frac{I^{n+1}M_i \colon x}{I^n M_i} \to \frac{\widetilde{I^n M_i}}{I^n M_i} \to \frac{\widetilde{I^{n+1} M_i}}{I^{n+1} M_i} \to \frac{\widetilde{I^{n+1}\overline{M_i}}}{I^{n+1}\overline{M_i}}.$$

By Lemma 4.1 we get that $\widetilde{I^n\overline{M_i}} = I^n\overline{M_i}$ for all $n \geq r$. By the above exact sequence we get that

$$\ell\left(\frac{\widetilde{I^r M_i}}{I^r M_i}\right) = \sum_{n\geq r} \ell\left(\frac{I^{n+1}M_i \colon x}{I^n M_i}\right) \leq \ell(B^I(M_i)), \text{and}$$

$$\ell\left(\frac{\widetilde{I^n M_i}}{I^n M_i}\right) \geq \ell\left(\frac{\widetilde{I^{n+1} M_i}}{I^{n+1} M_i}\right) \quad \text{for } n \geq r.$$

Claim: If $n \geq r$ and $\widetilde{I^n M_i} \neq I^n M_i$ then we have a strict inequality

$$\ell\left(\frac{\widetilde{I^n M_i}}{I^n M_i}\right) > \ell\left(\frac{\widetilde{I^{n+1} M_i}}{I^{n+1} M_i}\right)$$

Proof of Claim: If the result does not hold then we have $(I^{n+1}M_i \colon x) = I^n M_i$. For all $m \geq 1$ we have an exact sequence

$$\frac{(I^m M_i \colon x)}{I^{m-1}M_i} \xrightarrow{\beta_m} \frac{(I^{m+1}M_i \colon x)}{I^m M_i} \xrightarrow{\alpha_m} \frac{I^{m+1}M_i}{JI^m M_i} \xrightarrow{\rho_m} \frac{I^{m+1}\overline{M_i}}{yI^m\overline{\overline{M_i}}} \to 0,$$

(see proof of Lemma 5.3). So if $n \geq r$ and $(I^{n+1}M_i \colon x) = I^n M_i$ then $(I^{m+1}M_i \colon x) = I^m M_i$ for all $m \geq n$. So we get

$$\ell\left(\frac{\widetilde{I^n M_i}}{I^n M_i}\right) = \sum_{m\geq n} \ell\left(\frac{I^{m+1}M_i \colon x}{I^m M_i}\right) = 0, \quad \text{a contradiction.}$$

It follows that

$$\rho^I(M_i) \leq r + 1 + \ell(B^I(M_i)).$$

We consider the following two cases.

Case I: $\operatorname{depth} G_I(A) > 0$. So $x^*$ is $G_I(A)$-regular. By proof of Theorem 5.1 the function $i \mapsto \ell(B^I(M_i))$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$.

It follows that

$$\left\{\frac{\rho^I(M_i)}{i^{\operatorname{cx}(M)-1}}\right\}_{i\geq 1} \quad \text{is bounded.}$$

Case II. $\operatorname{depth} G_I(A) = 0$. We note that $\operatorname{depth} G_{I^m}(A) \geq 1$ for all $m \gg 0$. Choose $s$ such that $\operatorname{depth} G_{I^s}(A) \geq 1$. For $n \geq r$ we have

$$\ell\left(\frac{\widetilde{I^n M_i}}{I^n M_i}\right) \geq \ell\left(\frac{\widetilde{I^{n+1} M_i}}{I^{n+1} M_i}\right).$$

It follows that

$$\rho^I(M_i) \leq \max\{r, s\rho^{I^s}(M_i)\}.$$

By our Case I the result follows.

Now assume $d \geq 3$ and that the result holds when dimension of the ring is $= d-1$. Let $x$ be $M_i \oplus A$-superficial for all $i \geq 0$. Set $\overline{M_i} = M_i/xM_i$. Then notice $\overline{M}$ has finite GCI-dimension over $\overline{A}$ and $\overline{M_i} \cong \operatorname{Syz}_i^{\overline{A}}(\overline{M})$ for $i \geq 0$. We have exact sequences

$$0 \to U_i \to G_I(M_i)(-1) \xrightarrow{x^*} G_I(M_i) \to G_I(M_i)/x^* G_I(M_i) \to 0 \quad \text{and}$$
$$0 \to V_i \to G_I(M_i)/x^* G_I(M_i) \to G_I(\overline{M_i}) \to 0;$$

where $U_i, V_i$ are $G_I(A)$-modules of finite length. As $d \geq 3$ we have an exact sequence for all $n \in \mathbb{Z}$

$$H^{d-2}(G_I(\overline{M_i}))_n \to H^{d-1}(G_I(M_i))_{n-1} \to H^{d-1}(G_I(M_i))_n$$

It follows that

$$a_{d-1}(G_I(M_i)) \leq a_{d-2}(G_I(\overline{M_i})) - 1.$$

The result now holds by induction hypothesis. $\square$

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